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Sensitivity Analysis of Discrete Periodic Systems with Applications to Helicopter Rotor Dynamics

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This paper presents a sensitivity formulation for periodic systems and a theoretical model for a helicopter rotor in forward flight. The formulation and the rotor model are validated by a few comparisons with other available methods and experimental values. The greatest advantage of the present sensitivity analysis is that the order of the Floquet matrix is independent of the number parameters to be investigated. This yields substantial savings in computation times over the existing methods. The formulation yields unique derivatives for the eigenvalues in spite of the nonuniqueness in the eigenvalues, and this includes also the frequency-locked region. The derivatives of the eigenvectors also follow from the same formulation except that it breaks down in the frequency-locked region.

Nomenclature

a = lift-curve slope
B = tip loss factor
C_T = rotor thrust coefficient, $T/\rho\pi R^2(\Omega R)^2$
c = chord
c_d = airfoil drag coefficient
c_{d_a} = derivative of the drag coefficient with respect to lift-curve slope
e_{ac} = chordwise offset of aerodynamic center (ac) from the feathering axis, positive forward
e_{cg} = chordwise offset of center of gravity (cg) from the feathering axis, positive forward
e_{fa_x} = spanwise pitch bearing offset
e_{fa_y} = chordwise feathering axis offset (torque offset) from the center of rotation, positive forward
e_{rc} = blade root cutout
$I_\beta, I_\zeta, I_\theta$ = flap, lag, and pitch moments of inertia of the blade, respectively
k_m = mass polar radius of gyration of blade about the feathering axis
$k_\beta, k_\zeta, k_\theta$ = spring restraints of the blade for flap, lag, and pitch motions, respectively
m = mass of the blade
N = number of blades
p = system parameter
R = blade radius
T = time period or thrust
$[U]$ = modal matrix
V = forward speed
x, y, z = rotating coordinate system, x axis is along the blade feathering axis (positive forward), y axis is positive towards the leading edge, and z axis is positive up
α = rotor angle of attack
β_p = blade precone, positive up
γ = Lock number, $\rho acR^4/I_\beta$
ζ_s = blade presweep, positive lag
θ = blade pitch, $\theta_e + \theta_r$, positive nose up
θ_{1c}, θ_{1s} = lateral and longitudinal cyclic pitch components
θ_c = cyclic pitch control, $\theta_{1c} \cos \Psi + \theta_{1s} \sin \Psi$
θ_e = elastic twist
θ_r = blade rigid pitch, $\theta_0 + \theta_c + \theta_{tw}$
θ_{tw} = blade pretwist, positive noseup
θ_0 = collective pitch
λ = eigenvalue of the system
μ = advance ratio, $V \cos \alpha/\Omega R$
ξ_s = structural damping
ξ_ζ = critical damping coefficient in the lag motion
ρ = air density
σ = blade solidity, $Nc/\pi R$
Ψ = azimuth angle
Ω = angular velocity of rotor
$\omega_\beta, \omega_\zeta, \omega_\theta$ = nonrotating flap, lag, and torsion natural frequencies of the blade, respectively

Introduction

THE optimal design of engineering systems calls for the sensitivity derivatives with respect to the system design parameters. Thus, the sensitivity analysis plays an important role in the optimization process of aerospace systems, including structural and control system designs and system identification problems. Sensitivity analysis of constant coefficient systems with static, dynamic, and aeroelastic constraints has received considerable attention in the engineering literature during the last two decades.[1-5]

The derivatives of eigenvalues and eigenvectors are required in the sensitivity analyses of systems that are susceptible to stability problems, such as static, dynamic, and aeroelastic stabilities. These derivatives provide important information to the designer, allowing the designer to understand the constraints and methodically proceed with the optimization. These derivatives are also necessary in automated optimization procedures. The derivatives can be calculated using finite difference methods by simple repetitive solution of the eigenvalue problems. Because this repetitive approach is excessively time consuming, several direct analytical methods[6-13] are employed to calculate the derivatives of the eigenvalues and eigenvectors.

Rotary-wing aircraft are governed by the periodic equations of motion, and sensitivity studies[14-18] in this area are usually conducted using finite difference methods, since the methods developed for the constant systems are not applicable. Recently, Lim and Chopra[19-21] performed pioneering sensitivity studies in the area of rotary-wing dynamics. They employed a chain rule differentiation approach, which is more efficient than a finite difference approach, to calculate the derivatives of the eigenvalues. In the present paper, however, a direct analytical approach is developed to calculate the derivatives of the eigenvalues and eigenvectors of periodic systems. The greatest advantage of this method is that the order of the

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Floquet transition matrix that is requisite in obtaining the derivatives remains the same as in the original problem and independent of the number of parameters to be investigated. The derivatives of the eigenvectors are also generated from the same formulation. In the frequency-locked region, the derivative formulation for the eigenvalues is valid whereas it breaks down for the eigenvectors.

The formulation developed in this paper is applied to two problems in the rotor dynamics area. The first problem deals with the coupled rotor-body problem in hover. This problem provides an excellent means to validate the formulation and the computer program since it yields a periodic system in the individual coordinate system while yielding a constant system in the multiblade coordinates. The second problem deals with a helicopter rotor in forward flight. A rotor model is derived and the final equations are given in the Appendix. This model is validated by comparing the lag stability values with the experimental values, and the correlation found is excellent. A trim analysis of a helicopter in the time domain is performed and aeroelastic stability eigenvalues are computed about this trim state. The derivatives of the stability eigenvalues are then computed with respect to several parameters, such as precone, rotor angle of attack, torque offset, Lock number, thrust coefficient, and center of gravity (cg) offset. The results are compared with those obtained using the existing methods, such as chain rule differentiation and finite difference methods, and computation time savings in excess of 50% are observed.

The imaginary parts of the eigenvalues of periodic systems are nonunique to the extent of the integer multiples of 2π divided by the period, and the dominant eigenvalues have to be identified based on the harmonic contents of the principal eigenvector and on the continuity considerations when periodicity drops out in the limit. However, the derivatives of the eigenvalues are unique, since the nonuniqueness in the modal and its biorthogonal matrices employed in the formulation negate each other. The eigenvectors associated with the nonunique eigenvalues are proportional to each other in modulus form. These aspects of the periodic systems are numerically verified by applying the present formulation to an isolated flapping blade in forward flight.

Formulation

Consider a linear and periodic system described by homogeneous and ordinary differential equation of the form

$$\{\dot{X}(t)\} = [A(t)]\{X(t)\} \tag{1}$$

where $[A(t)]$ is a square periodic system matrix of order n, with period T, and $\{X(t)\}$ is a state vector of the system. The solution of this equation is of the form

$$\{X(t)\} = [\Phi(t,0)]\{X(0)\} \tag{2}$$

where $[\Phi(t,0)]$ is the state transition matrix from 0 to t.

The stability of periodic systems can be examined by the Floquet-Liapunov theory,[22] which states that the transition matrix of a periodic system is of the form

$$[\Phi(t)] = [P(t)]e^{[\beta]t} \tag{3}$$

where $[P(t)]$ is a periodic matrix and $[\beta]$ is a constant matrix. Form Eq. (3), one can write the Floquet transition matrix (which is simply the transition matrix for a single period) as

$$[\alpha] = [\Phi(T)] = e^{[\beta]T} \tag{4}$$

Equation (4) results from Eq. (3) following from $[\Phi(0)] = [{}^{\diagdown}1_{\diagdown}]$, where $[{}^{\diagdown}1_{\diagdown}]$ is an identity matrix and $[P(t)]$ is periodic. From the product rule of transition matrices that states $[\Phi(t_2,0)] = [\Phi(t_2,t_1)][\Phi(t_1,0)]$, and from Eqs. (3) and (4), one can write that

$$[\Phi(t+mT)] = [\Phi(t)][\alpha]^m, \quad m = 0, 1, 2, \ldots, . \tag{5}$$

If $[{}^{\diagdown}\lambda_{\diagdown}]$ and $[S]$ are the eigenvalue and modal matrices of $[\beta]$, it follows from the matrix theory that

$$[S]^{-1}[\beta][S] = [{}^{\diagdown}\lambda_{\diagdown}] \tag{6}$$

$$[S]^{-1}e^{[\beta]t}[S] = [{}^{\diagdown}e^{\lambda t}{}_{\diagdown}] \tag{7}$$

Since $[\alpha] = e^{[\beta]T}$ from Eq. (4), it follows then from Eq. (7) that $[S]$ is also a modal matrix of Floquet transition matrix $[\alpha]$. The eigenvalues of the Floquet transition matrix are then specified by

$$[S]^{-1}[\alpha][S] = [{}^{\diagdown}\Lambda_{\diagdown}] \tag{8}$$

Therefore, it follows from Eqs. (7) and (8) that

$$[S]^{-1}[\alpha]^m[S] = [{}^{\diagdown}\Lambda^m{}_{\diagdown}] \tag{9}$$

$$\lambda = \frac{1}{T}\ln\Lambda \tag{10}$$

From Eqs. (2), (5), (9), and (10), one can conclude that the solution is unstable if either $|\Lambda| > 1$ or $\mathrm{Re}\lambda > 0$. Since the logarithm of a complex variable is a multivalued function, Eq. (10) can be written as a principal part plus integer multiples of $(2\pi i/T)$ as shown next:

$$\lambda = (1/T)\left[\ln|\Lambda| + i\angle\Lambda\right] + (2m\pi i/T) \tag{11}$$

where m is any integer and

$$|\Lambda| = \sqrt{(\mathrm{Re}\Lambda)^2 + (\mathrm{Im}\Lambda)^2}$$

$$\angle\Lambda = \arctan\,(\mathrm{Im}\,\Lambda/\mathrm{Re}\,\Lambda)$$

Substituting Eqs. (3) and (7) into the transient response given by Eq. (2) yields

$$\{X(t)\} = [P(t)][S][\ e^{\lambda t}\ ][S]^{-1}\{X(0)\} \tag{12}$$

The preceding equation can be written as

$$\{X(t)\} = [U(t)]\{q(t)\} \tag{13}$$

where $[U(t)] = [P(t)][S]$ is considered as a periodic modal matrix of the dynamic system and $\{q(t)\} = [{}^{\diagdown}e^{\lambda t}{}_{\diagdown}]\{q(0)\}$ are considered as modal coordinates. The initial conditions for the modal coordinates are given by $\{q(0)\} = [S]^{-1}\{X(0)\}$. Equation (13) can be written, in terms of the eigenvectors, as

$$\{X(t)\} = \sum_{i=1}^{n}\{u_i(t)\}e^{\lambda_i t}q_i(0) \tag{14}$$

Substituting Eqs. (2) and (3) into Eq. (1) yields

$$[\dot{P}] = [A][P] - [P][\beta] \tag{15}$$

Substituting Eq. (6) into the preceding equation results in

$$[\dot{P}][S] = [A][P][S] - [P][S][{}^{\diagdown}\lambda_{\diagdown}] \tag{16}$$

The differential equation governing the eigenvector $\{u_k(t)\}$ can then be derived from Eq. (16) as

$$\{\dot{u}_k\} = ([A] - \lambda_k[{}^{\diagdown}1_{\diagdown}])\{u_k\} \tag{17a}$$

The homogeneous boundary condition for this eigenvalue problem is the periodic condition given by

$$\{u_k(t)\} = \{u_k(t+T)\} \tag{17b}$$

The first-order variation of Eq. (17a) is given by

$$\delta\{\dot{u}_k(t)\} = \left(\delta[A(t)] - \delta\lambda_k [{}^{\diagdown}1_{\diagdown}]\right)\{u_k(t)\} + \left([A(t)] - \lambda_k [{}^{\diagdown}1_{\diagdown}]\right)\delta\{u_k(t)\} \tag{18}$$

Premultiplying Eq. (18) by the transpose of the biorthogonal vector $\{v_k(t)\}$ of $\{u_k(t)\}$ yields

$$\{v_k\}^T\delta\{\dot{u}_k\} = \{v_k\}^T\left(\delta[A] - \delta\lambda_k [{}^{\diagdown}1_{\diagdown}]\right)\{u_k\} + \{v_k\}^T\left([A] - \lambda_k [{}^{\diagdown}1_{\diagdown}]\right)\delta\{u_k\} \tag{19}$$

The biorthogonal matrix $[V(t)]$ of $[U(t)]$ satisfies

$$[V(t)]^T[U(t)] = [{}^{\diagdown}1_{\diagdown}] \tag{20}$$

The variation of the eigenvector can be expanded as

$$\delta\{u_k(t)\} = \sum_{i=1}^{n} \delta\eta_{ik}(t)\{u_i(t)\} \tag{21}$$

where $\delta\eta_{ik}(t)$ is a periodic function of t since both $\delta\{u_k(t)\}$ and $\{u_i(t)\}$ are periodic functions. The substitution of Eq. (21) into Eq. (19) yields, after simplification,

$$\delta\lambda_k = -\delta\dot{\eta}_{kk} + \{v_k\}^T\delta[A]\{u_k\} \tag{22a}$$

or

$$\frac{\partial\lambda_k}{\partial p} \approx -\frac{\partial\dot{\eta}_{kk}}{\partial p} + \{v_k\}^T\frac{\partial[A]}{\partial p}\{u_k\} \tag{22b}$$

The derivative of eigenvalue λ_k, with respect to a system parameter p, can be defined as

$$\frac{\mathrm{d}s\lambda_k}{\mathrm{d}p} = \frac{1}{T}\int_0^T \frac{\partial\lambda_k}{\partial p}\,\mathrm{d}\Psi \tag{23}$$

Substituting Eq. (22b) into Eq. (23) yields

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}p} = \frac{1}{T}\int_0^T \{v_k(t)\}^T\left(\frac{\mathrm{d}[A(t)]}{\mathrm{d}p}\right)\{u_k(t)\}\,\mathrm{d}t \tag{24}$$

for the derivative of eigenvalue λ_k of a periodic system.

Premultiplying Eq. (18) by $\{v_j(t)\}^T$ $(j \neq k)$ yields

$$\{v_j\}^T\delta\{\dot{u}_k\} = \{v_j\}^T\delta[A]\{u_k\} + \{v_j\}^T[A]\delta\{u_k\} - \lambda_K\{v_j\}^T\delta\{u_k\} \tag{25}$$

Substituting Eq. (21) into Eq. (25) and simplifying the resulting equation yields

$$\delta\dot{\eta}_{jk}(t) + (\lambda_k - \lambda_j)\delta\eta_{jk}(t) = \{v_j\}^T\delta[A(t)]\{u_k(t)\}, \quad j \neq k \tag{26}$$

The derivative of this equation, with respect to a system parameter p, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\eta_{jk}(t)}{\mathrm{d}p}\right) + (\lambda_k - \lambda_j)\frac{\mathrm{d}\eta_{jk}(t)}{\mathrm{d}p} = \{v_j\}^T\left(\frac{\mathrm{d}[A(t)]}{\mathrm{d}p}\right)\{u_k(t)\} \tag{27a}$$

$$j, k = 1, 2, \ldots, n, \text{ and } j \neq k.$$

The boundary condition for this differential equation is the periodicity condition given by

$$\frac{\mathrm{d}\eta_{kk}(t)}{\mathrm{d}p} = \frac{\mathrm{d}\eta_{jk}(t+T)}{\mathrm{d}p}, \qquad j \neq k \tag{27b}$$

Table 1 Eigenvalues of coupled rotor-body problem

Periodic system		Constant system	
Re	Im	Re	Im
-0.3125	± 1.0547	-0.3125	± 1.0547
-0.3058	± 2.0548	-0.3057	± 2.0549
-0.1668	± 0.2172	-0.1674	± 0.2207
-0.1524	± 0.2935	-0.1519	± 0.2942

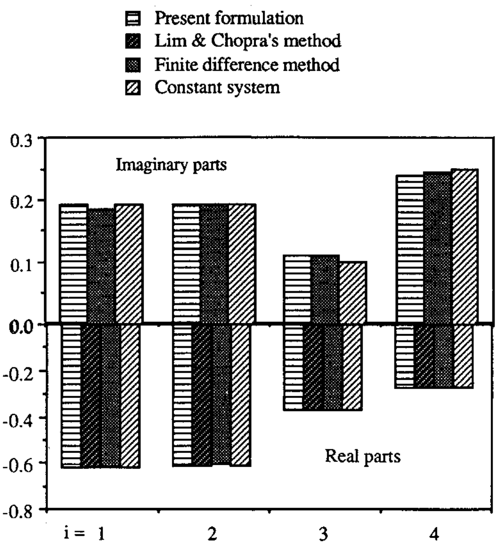


Fig. 1 Derivatives of eigenvalues of coupled rotor-body problem, $\mathrm{d}\lambda_i/\mathrm{d}\gamma$.

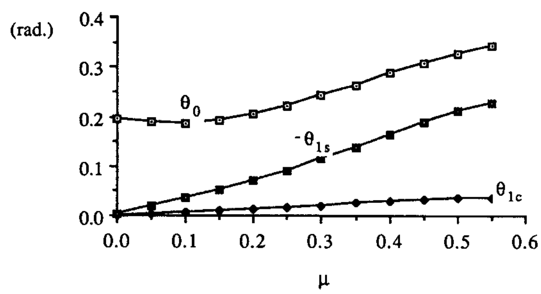


Fig. 2 Trim control angles.

The $\mathrm{d}\eta_{kk}/\mathrm{d}p$ can be determined using the normalizing condition of the eigenvector shown as

$$\int_0^T \{u_k(t)\}^T\{u_k(t)\}\,\mathrm{d}t = 1 \tag{28}$$

The first-order variation of Eq. (28) is given by

$$\int_0^T \{u_k(t)\}^T\delta\{u_k(t)\}\mathrm{d}t = 0 \tag{29}$$

Substituting Eq. (21) into the preceding equation yields

$$\frac{\mathrm{d}\eta_{kk}}{\mathrm{d}p} = -\sum_{\substack{i=1 \\ i\neq k}}^{n}\int_0^T \{u_k(t)\}^T\frac{\mathrm{d}\eta_{ki}(t)}{\mathrm{d}p}\{u_i(t)\}\,\mathrm{d}t \tag{30}$$

Finally, the derivative of the modal matrix with respect to a system parameter p follows from Eq. (21) as

$$\frac{\mathrm{d}[U(t)]}{\mathrm{d}p} = [U(t)]\frac{\mathrm{d}[\eta]}{\mathrm{d}p} \tag{31}$$

where the matrix $\mathrm{d}[\eta]/\mathrm{d}p$ can be computed from Eqs. (27) and (30).

Table 2 Rotor data for aeroelastic stability

$B = 1.0$	$c/R = 0.055$	$\omega_\theta = 3.03237$ 1/rev
$a = 5.7$	$c_d = 0.01$	$\omega_\zeta = 0.704$ 1/rev
$C_T = 0.005$	$e_{rc}/R = 0.05$	$\omega_\beta = 0.5132$ 1/rev
$\sigma = 0.07$	$\zeta_s = 0.0$	$k_m/R = 0.01295$
$\mu = 0.3$	$e_{fa_y} = 0$	$k_\theta/\Omega^2 I_\theta = 9.1953$ (1/rev)^2
$N = 4$	$R = 16.11$ ft	$k_\beta/\Omega^2 I_\beta = 0.2634$ (1/rev)^2
$\alpha = 10$ deg	$\beta_p = 0$	$k_\zeta/\Omega^2 I_\beta = 0.4956$ (1/rev)^2
$\gamma = 5.0$	$\xi_s = 0.03$	$e_{cg}/R = -0.00006$
$e_{fa_x} = 0$	$e_{ac} = 0$	$\theta_{tw} = -2.25$ deg

Table 3 Eigenvalues of trimmed rotor in forward flight

	Flap		Pitch		Lag	
μ	Re	Im	Re	Im	Re	Im
0.00	-0.3705	± 1.0770	-0.2284	± 3.1680	-0.03005	± 0.6798
0.10	-0.3715	± 1.1070	-0.2283	± 3.1670	-0.02916	± 0.6847
0.20	-0.3691	± 1.1410	-0.2282	± 3.1670	-0.03166	± 0.6826
0.30	-0.3642	± 1.1750	-0.2282	± 3.1680	-0.03659	± 0.6772
0.40	-0.3572	± 1.2100	-0.2282	± 3.1680	-0.04355	± 0.6711
0.50	-0.3502	± 1.2430	-0.2281	± 3.1680	-0.05070	± 0.6672
0.55	-0.3482	± 1.2570	-0.2280	± 3.1680	-0.05271	± 0.6672

Table 4 Eigenvalues of an isolated flapping blade in forward flight

	Ψ	$\lvert U_{11}\rvert$	$\lvert U_{12}\rvert$	$\lvert U_{21}\rvert$	$\lvert U_{22}\rvert$
$[U(\lambda_1)]$	0.00000	0.31246	0.31246	0.49625	0.49625
	1.25660	0.28770	0.28770	0.26737	0.26737
	2.51330	0.37784	0.37784	0.26203	0.26203
	3.76990	0.15251	0.15251	0.38446	0.38446
	5.02650	0.36839	0.36839	0.19917	0.19917
	6.28320	0.31246	0.31246	0.49625	0.49625
$[U(\lambda_2)]$	0.00000	0.54536	0.54536	0.86615	0.86615
	1.25660	0.50215	0.50215	0.46667	0.46667
	2.51330	0.65947	0.65947	0.45734	0.45734
	3.76990	0.26618	0.26618	0.67102	0.67102
	5.02650	0.64299	0.64299	0.34762	0.34762
	6.28320	0.54536	0.54536	0.86615	0.86615

[a]$\gamma = 6$, $k\beta/\Omega^2 I\beta = 0.21$, $\mu = 0.3$
$\lambda_1 = -0.375 \pm 0.0162i$, principal eigenvalue
$\lambda_2 = -0.375 \pm 1.0162i$, dominant eigenvalue
$\lambda = -0.375 \pm 1.0341i$, $\mu = 0.0$

Applications

Two helicopter dynamics problems are considered for application of the formulation developed. The first problem deals with a coupled rotor body in hover that yields a periodic system in the individual blade coordinates and a constant system in the multiblade coordinates. This problem provides an excellent means to validate the formulation. The helicopter body is assumed to be rigid with pitch and roll degrees of freedom. A three-bladed rotor with flapping motion restrained by linear elastic springs, with quasisteady aerodynamics, is used in the derivation of the equations of motion. The resulting equations are given by

$$[M(\Psi)]\{\ddot{X}(\Psi)\} + [C(\Psi)]\{\dot{X}(\Psi)\} + [K(\Psi)]\{X(\Psi)\} = \{0\} \tag{32}$$

where $\{X(\Psi)\} \equiv (\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \alpha_x, \alpha_y)^T$ consists of individual blade and body pitch and roll coordinates, and $[M(\Psi)]$, $[C(\Psi)]$, and $[K(\Psi)]$ are periodic coefficient matrices. The periodicity in the previous equation can be eliminated by employing the following multiblade coordinate transformation:

$$\beta_0 = \frac{1}{3}\sum_{j=1}^{3}\beta^{(j)}, \qquad \beta_{1c} = \frac{2}{3}\sum_{j=1}^{3}\beta^{(j)}\cos\Psi_j$$

$$\beta_{1s} = \frac{2}{3}\sum_{j=1}^{3}\beta^{(j)}\sin\Psi_j, \qquad \Psi_j = \Psi + \frac{2\pi}{3}j$$

The resulting equation in the multiblade coordinates is given by

$$[M]^*\{\ddot{X}^*(\Psi)\} + [C]^*\{\dot{X}^*(\Psi)\} + [K]^*\{X^*(\Psi)\} = \{0\} \tag{33}$$

where $\{X^*\} = (\beta_0, \beta_{1c}, \beta_{1s}, \alpha_x, \alpha_y)^T$. The model and the coefficient matrices of the equations of motion are listed in the Appendix. Equations (32) and (33) have to be reduced to the following form to apply the present sensitivity formulation:

$$\{\dot{Y}\} = [A]\{Y\} \tag{34}$$

where

$$\{Y\} = (\{\dot{X}\}, \{X\})^T$$

and

$$[A(\Psi)] = \begin{bmatrix} [0] & [\diagdown 1 \diagdown] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}$$

The second problem deals with a four-bladed rotor in forward flight. Each blade is assumed to be rigid with spring restraints, and flap, lag, and pitch degrees of freedom are considered in the analysis. The following equation is derived for the blade motion with a quasisteady aerodynamic theory:

$$[B]\{\dot{X}(\Psi)\} = [A(\Psi)]\{X(\Psi)\} + \{f(\Psi)\} + \{F(\Psi, \{X\}, \{\dot{X}\})\} \tag{35}$$

where $\{X(\Psi)\} \equiv \{\dot{\beta}, \beta, \dot{\zeta}, \zeta, \dot{\theta}_e, \theta_e\}^T$, $[A]$ is a periodic coefficient matrix, and $\{f\}$ and $\{F\}$ are nonhomogeneous and nonlinear forcing functions. The model description and the coefficient matrices are also listed in the Appendix.

The rotor is trimmed for the given weight ($T = W$) and for zero steady-state hub rolling and pitching moments ($M_x = M_y = 0$). The blade equation of motion [Eq. (35)] is integrated in the time domain, and trim control settings are determined using a Newton-Raphson interactive procedure. The blade perturbation equation about the trim state is

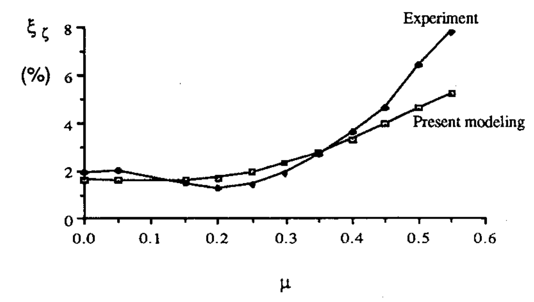


Fig. 3 Comparison of lag stability with experimental values.

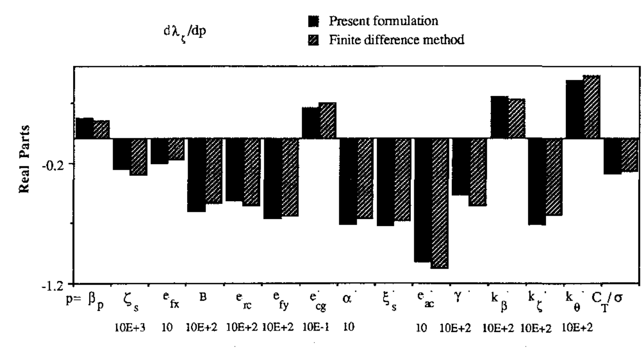


Fig. 4 Lag damping derivatives.

obtained substituting $\Delta\{X(\Psi)\} = \{X(\Psi)\} - \{X_0(\Psi)\}$ into Eq. (35) where $\{X_0(\Psi)\}$ is the trimmed state vector. The resulting perturbation equation is given by

$$\Delta\{\dot{X}\} = [D(\Psi)]\Delta\{X\} \tag{36}$$

where

$$[D(\Psi)] = \left([B] + \left[B(\{X_0\})\right]^*\right)^{-1}\left([A(\Psi)] + \left[A(\Psi,\{X_0\})\right]^*\right) \tag{37}$$

The elements of matrices $[A]^*$ and $[B]^*$ are also listed in the Appendix, and the matrices $[A]$ and $[B]$ are the same as those given in Eq. (35). The aeroelastic stability and the sensitivity studies can now be performed applying the present formulation to Eq. (36).

Numerical Results

The eigenvalues of the periodic coupled rotor-body problem described by Eq. (32) are computed and presented in Table 1 and are then compared with the equivalent constant system results obtained by solving Eq. (33). The numerical results presented in this table correspond to system parameters given by $\gamma = 5$, $\nu_\beta = 1.1$, $I_p = 3.0$, $I_q = 4.0$. The derivatives of eigenvalues with respect to Lock number (γ) are presented in Fig. 1. The derivatives obtained using this formulation are compared with the results obtained using Lim and Chopra's method,[20] the finite difference method, and the equivalent constant system in the same figure. The periodic system is used for the first three methods whereas a standard procedure, such as those described in Ref. 1, is used for the constant system results. The excellent agreement between the results obtained using the various methods clearly validates the present formulation. Not included in this paper for the sake of brevity is the excellent agreement between the derivatives of eigenvectors obtained using both the present formulation and the finite difference method.

The rotor data employed for the calculation of aeroelastic stability derivatives in forward flight is shown in Table 2. The trim control settings for advance ratios up to 0.55 are plotted in Fig. 2. The stability eigenvalues about the trim states are given in Table 3 for the same advance ratios. The numerical data employed in the present calculations are very close to the MBB-105 helicopter rotor data presented in Ref. 23. A comparison of the lag stability with the experimental values is shown in Fig. 3, and an excellent correlation can be observed from this figure. Since the lag motion dominates the aeroelastic stability of modern rotors, the derivatives of the lag stability eigenvalues with respect to several parameters are computed and presented in Fig. 4 together with the results of finite difference solutions. Agreement is excellent.

Lim and Chopra[21] studied the sensitivity of periodic systems employing an analytical procedure (chain rule differentiation) that is more efficient than a finite difference method. However, the size of the system matrix in their procedure increases by an amount equal to the number of parameters, multiplied by the number of degrees of freedom of the original problem. The size of the matrix differential equation to be solved under the present formulation is independent of the number of parameters and remains the same as in the original problem. For instance, a 3 degree-of-freedom problem with 10 parameters will result in a 66×66 system if Lim and Chopra's procedure is employed but will remain as 6×6 under the present formulation. The derivatives of eigenvalues of the coupled rotor-body problem are calculated using the present formulation, Lim and Chopra's procedure, and a finite difference method. The CPU times required for these three methods on a VAX/8820 computer are plotted in Fig. 5. We can see from this figure that the present formulation for a problem of four parameters takes only 55% of the time necessary when using Lim and Chopra's procedure and only 40% of the time needed to employ the finite difference method. This efficiency of the present method increases substantially with the number of parameters, as shown in Fig. 6.

The eigenvalues and the associated eigenvectors of an isolated flapping blade are calculated and presented in Table 4 in modulus form. It is to be noted from Eq. (11) that the eigenvalues are not unique but can differ from one another by integer multiples of $2\pi i/T$. This does not make any difference in the unique determination of the solution of the problem. From Eq. (14), it can be seen that the actual response of the system depends on the product $\{u(t)\}e^{\lambda t}$. If an integer multiple of the fundamental frequency ($2\pi i/T$) is added to the eigenvalue (λ), then to keep the response product $\{u(t)\}e^{\lambda t}$ unchanged, the eigenvector $\{u(t)\}$ should be multiplied by the periodic function $e^{-2\pi n i/T}$. From Eq. (17b), one can see that the Floquet theory requires only the eigenvector $\{u(t)\}$ be periodic but never restricts the allotment of periodicity between the eigenvalue and its eigenvector. From the expansion of $\{u(t)\}e^{\lambda t}$ as a Fourier series,

$$\{u(t)\}e^{\lambda t} = \sum_{n=-\infty}^{n=\infty} \{u_n\}e^{(\lambda + in2\pi/T)}$$

it follows that the natural vibration of the periodic system contains the principal eigenvalue λ plus or minus all integer multiples of $2\pi i/T$. The Fourier coefficients u_n in this equation give the information as to how much of each harmonic will occur in the total natural vibration. The dominant frequency in the natural vibration is usually identified by the frequency content of the eigenvector. The principal eigenvalue λ_p and the associated principal eigenvector can be determined uniquely, and the harmonic of largest magnitude in this eigenvector gives the dominant frequency. In the helicopter problems, the dominant frequency can usually be determined also from the requirement that the roots be continuous as the periodicity drops out in the limit $\mu = 0$. In the present case, the eigenvalue $\lambda_2 = -0.375 \pm i0.0162$ is identified as the dominant frequency in the preceding fashion. Also shown in Table 4 is another eigenvector of the system corresponding to the principal eigenvalue $\lambda_1 = -0.375 \pm i0.162$. It is to be noted that λ_1

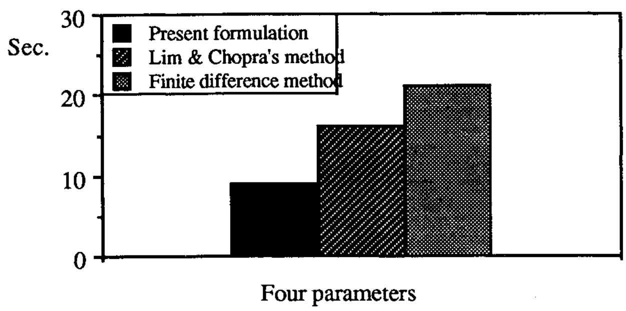


Fig. 5 Comparison of CPU time for coupled rotor-body problem.

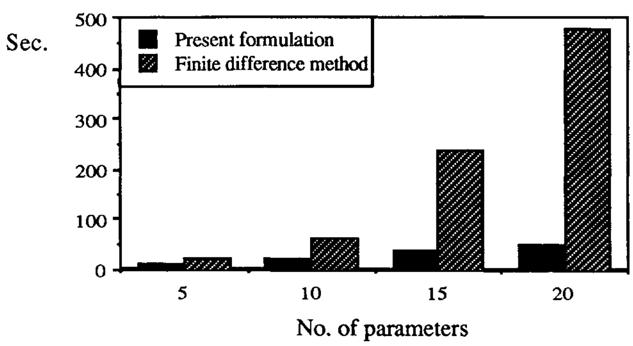


Fig. 6 Comparison of CPU time for forward flight rotor problem.

Table 5 Derivatives of eigenvectors

	Ψ	$\left\vert\frac{dU_{11}}{dp}\right\vert$	$\left\vert\frac{dU_{12}}{dp}\right\vert$	$\left\vert\frac{dU_{21}}{dp}\right\vert$	$\left\vert\frac{dU_{22}}{dp}\right\vert$
$[U(\lambda_1)]$	0.00000	0.38410	0.38410	0.26413	0.26413
$p=\gamma$	1.25660	0.34667	0.34667	0.45828	0.45828
	2.51330	0.16259	0.16259	0.31510	0.31510
	3.76990	0.36068	0.36068	0.14443	0.14443
	5.02650	0.23840	0.23840	0.46860	0.46860
	6.28320	0.38410	0.38410	0.26413	0.26413
$p=k_\beta$	0.00000	12.8030	12.8030	9.39850	9.39850
	1.25660	12.0830	12.0830	15.4260	15.4260
	2.51330	5.32710	5.32710	11.0920	11.0920
	3.76990	12.4460	12.4460	5.25630	5.25630
	5.02650	7.60880	7.60880	16.1360	16.1360
	6.28320	12.8030	12.8030	9.39850	9.39850
$p=\mu$	0.00000	2.74580	2.74580	1.74170	1.74170
	1.25660	2.27490	2.27490	3.28840	3.28840
	2.51330	1.09720	1.09720	2.21320	2.21320
	3.76990	2.49480	2.49840	1.02620	1.06620
	5.02650	1.81930	1.81930	3.19620	3.19620
	6.28320	2.74580	2.74580	1.74170	1.74170
$[U(\lambda_2)]$	0.00000	0.62528	0.62528	0.51050	0.51050
$p=\gamma$	1.25660	0.36931	0.36931	0.64083	0.64083
	2.51330	0.49207	0.49207	0.36826	0.36826
	3.76990	0.46511	0.46511	0.41762	0.41762
	5.02650	0.35946	0.35946	0.60144	0.60144
	6.28320	0.62528	0.62528	0.51050	0.51050
$p=k_\beta$	0.00000	21.8990	21.8990	20.3430	20.3430
	1.25660	13.4700	13.4700	22.2910	22.2910
	2.51330	18.8120	18.8120	12.6240	12.6240
	3.76990	15.3610	15.3610	17.7050	17.7050
	5.02650	15.2200	15.2200	20.1370	20.1370
	6.28320	21.8990	21.8990	20.3430	20.3430
$p=\mu$	0.00000	4.58300	4.58300	4.24090	4.24090
	1.25660	2.59890	2.59890	4.60600	4.60600
	2.51330	3.99080	3.99080	2.61990	2.61990
	3.76990	3.10190	3.10190	3.55210	3.55210
	5.02650	3.18610	3.18610	4.05280	4.05280
	6.28320	4.58300	4.58300	4.24090	4.24090

Table 6 Comparison of derivatives of eigenvectors

	Ψ	$\left\vert\frac{dU_{11}}{d\gamma}\right\vert$	$\left\vert\frac{dU_{12}}{d\gamma}\right\vert$	$\left\vert\frac{dU_{21}}{d\gamma}\right\vert$	$\left\vert\frac{dU_{22}}{d\gamma}\right\vert$
Present	0.00000	0.62528	0.62528	0.51050	0.51050
formulation	1.25660	0.36931	0.36931	0.64083	0.64083
	2.51330	0.49207	0.49207	0.36826	0.36826
	3.76990	0.46511	0.46511	0.41762	0.41762
	5.02650	0.35946	0.35946	0.60144	0.60144
	6.28320	0.62528	0.62528	0.51050	0.51050
Finite	0.00000	0.62351	0.62351	0.59080	0.59080
difference	1.25660	0.41766	0.41766	0.59740	0.59740
method	2.51330	0.50142	0.50142	0.39344	0.39344
	3.76990	0.47716	0.47716	0.46911	0.46911
	5.02650	0.39875	0.39875	0.58951	0.58951
	6.28320	0.62351	0.62351	0.59080	0.59080

and λ_2 differ by $2n\pi i/T$ between $n=1$ and $T=2\pi$. From this table one can see that model matrices $[U(\lambda_1)]$ and $[U(\lambda_2)]$ are proportional to each other since the absolute values are given in this table. Strictly speaking, they should differ by a periodic factor e^{in}, but in absolute form this factor becomes unity. Even though the eigenvalues are not unique, the derivatives of eigenvalues given by Eq. (24) are unique since the nonuniqueness of $\{u_k\}$ and $\{v_k\}$ is negated in this equation.

The derivatives of both the eigenvectors shown in Table 4 are given in Table 5 with respect to the system parameters γ, μ, and k_β. From the results shown in this table, it can be observed that $d[U(\lambda_1)]/dp \neq d[U(\lambda_2)]/dp$, since they correspond to two different modes of motion. Finally, a comparison of the derivatives of the actual eigenvectors with those of the finite difference method is presented in Table 6, showing excellent agreement.

For periodic systems three types of principal eigenvalues can result: a complex conjugate pair, or a single real root, or a root with a one-half per revolution imaginary part. It is to be noted that the integer multiples $i2\pi/T$ can always be added to the principal eigenvalues for periodic systems. This can give rise to a frequency-locked instability at a frequency that is an integer multiple of one-half of the fundamental frequency of the system $(2\pi/T)$. The eigenvalues of a periodic system are continuous functions of the system parameters irrespective of the regions to which they belong, including the frequency-locked region. The derivative of the eigenvalue developed in this paper as given by Eq. (24) is valid in all regions, including the frequency-locked region. However, the regular perturbation expansion of the eigenvector variation breaks down when the frequency lock occurs. The authors plan to publish the details of this analysis in a follow-up paper.

Conclusions

1) An analytical formulation is developed to determine the derivatives of eigenvalues and eigenvectors of periodic systems. The formulation is very efficient because the size of the Floquet transition matrix that is computed to determine the derivatives does not depend on the number of parameters to be investigated. The derivatives of eigenvectors also follow from the same formulation.

2) The formulation is validated when applied to a periodic coupled rotor-body problem that has an equivalent constant system in terms of multiblade coordinates. The agreement with the constant system results is excellent.

3) A flap-lag-torsion model of a blade in forward flight is presented in the Appendix, and the lag stability obtained from this model is validated in the comparison with the experimental values. The derivatives of eigenvalues and eigenvectors for a rotor in forward flight are computed based on this model. The derivatives are compared with the results obtained using two existing methods. It is obvious then that the present formulation saves substantial computer time compared with the two established methods.

4) The eigenvalues of the system are nonunique to the extent of integer multiples of $2\pi i/T$; however, the derivatives of the eigenvalues are unique regardless of the nonuniqueness of the eigenvalues.

5) In the frequency-locked region, the formulation to determine the derivatives of the eigenvalues is valid whereas it breaks down for the eigenvectors.

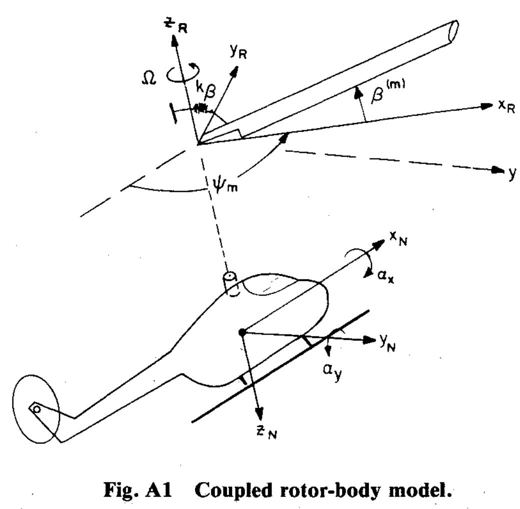


Fig. A1 Coupled rotor-body model.

Appendix

Equations of Motion of a Coupled Rotor-Body in Hover

Equations of motion for the model shown in Fig. A1 in terms of individual blade coordinates can be derived as

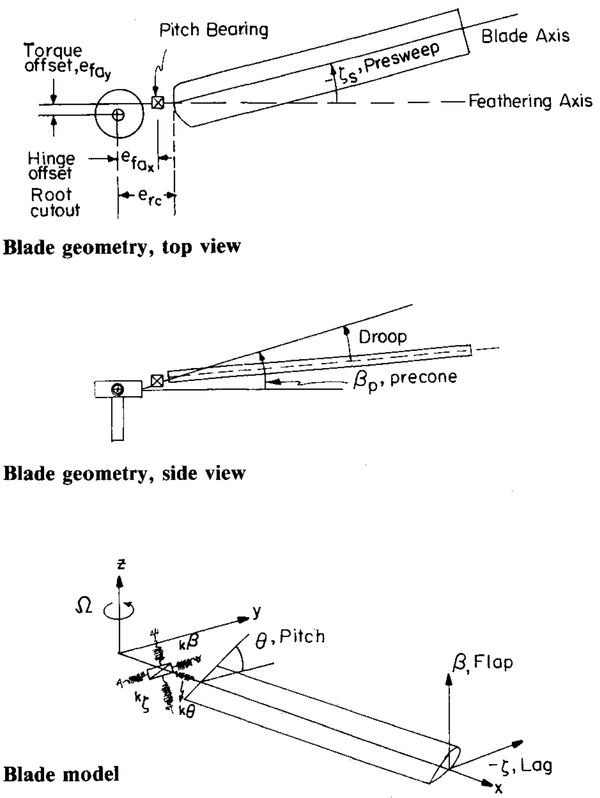
$$[M(\Psi)]\{\ddot{X}(\Psi)\} + [C(\Psi)]\{\dot{X}(\Psi)\} + [K(\Psi)]\{X(\Psi)\} = \{0\} \tag{A1}$$

where $\{X(\Psi)\} \equiv [\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \alpha_x, \alpha_y]^T$

$$[M(\Psi)] = \begin{bmatrix} 1 & 0 & 0 & S_1 & C_1 \\ 0 & 1 & 0 & S_2 & C_2 \\ 0 & 0 & 1 & S_3 & C_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K(\Psi)] = \begin{bmatrix} \nu_\beta^2 & 0 & 0 & 0 & 0 \\ 0 & \nu_\beta^2 & 0 & 0 & 0 \\ 0 & 0 & \nu_\beta^2 & 0 & 0 \\ aS_1 & aS_2 & aS_3 & 0 & 0 \\ bS_1 & bS_2 & bS_3 & 0 & 0 \end{bmatrix}$$

$$[C(\Psi)] = \begin{bmatrix} \gamma/8 & 0 & 0 & 2C_1 + S_1\gamma/8 & 2S_1 - C_1\gamma/8 \\ 0 & \gamma/8 & 0 & 2C_2 + S_2\gamma/8 & 2S_2 - C_2\gamma/8 \\ 0 & 0 & \gamma/8 & 2C_3 + S_3\gamma/8 & 2S_3 - C_3\gamma/8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



a) Blade geometry, top view

b) Blade geometry, side view

c) Blade model

Fig. A2 Rotor geometry and model.

where

$C_m = \cos \Psi_m$
$\Delta\Psi = 2\pi/N$
$\nu_\beta^2 = \omega_\beta^2 + 1$
$S_m = \sin \Psi_m$
$N = 3$
$\omega_\beta^2 = k_\beta/I_\beta$
$\Psi_m = \Psi + m\Delta\Psi$
$m = 1, 2, 3$
$a = -\omega_\beta^2/I_p$
$b = \omega_\beta^2/I_q$
I_β = flapping moment inertia of the blade
I_p = rolling moment of inertia of the body/I_β
I_q = pitching moment inertia of the body/I_β

The preceding equation for the coupled rotor-body problem [Eq. (A1)] can be expressed in terms of the multiblade coordinates as

$$[M]^*\{\ddot{X}^*(\Psi)\} + [C]^*\{\dot{X}^*(\Psi)\} + [K]^*\{X^*(\Psi)\} = \{0\} \tag{A2}$$

where $\{X^*\} = (\beta_0, \beta_{1c}, \beta_{1s}, \alpha_x, \alpha_y)^T$ and

$$[M]^* = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & -3/2 \\ 0 & 0 & 3/2 & 3/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K]^* = \begin{bmatrix} 3\nu_\beta^2 & 0 & 0 & 0 & 0 \\ 0 & 3\omega_\beta^2/2 & 3\gamma/16 & 0 & 0 \\ 0 & -3\gamma/16 & 3\omega_\beta^2/2 & 0 & 0 \\ 0 & 0 & 3a/2 & 0 & 0 \\ 0 & 3b/2 & 0 & 0 & 0 \end{bmatrix}$$

$$[C]^* = \begin{bmatrix} 3\gamma/8 & 0 & 0 & 0 & 0 \\ 0 & 3\gamma/16 & 3 & 3 & -3\gamma/16 \\ 0 & -3 & 3\gamma/16 & 3\gamma/16 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equations of Motion of a Rotor Blade in Forward Flight

The model employed in the derivation of equations of motion of a rotor blade in forward flight is shown in Fig. A2. The following assumptions are made in the derivation.

1) Rigid blade with linear elastic restraints k_β (flap), k_θ (pitch), and K_ζ (lag) is assumed.

2) Pitch bearing and virtual or real flap and lag hinges are assumed to lie at the same location e_{fa_x} from the root; however, flap (β), pitch (θ), and lag (ζ) hinge sequences are assumed to progress from the undeformed coordinate system to the deformed coordinate system.

3) The blade is assumed to have torque offset (e_{fa_y}, positive as shown), precone (β_p, positive up), presweep (ζ_s, positive lag), pretwist (θ_{tw}, positive nose up), and zero droop.

4) Quasisteady aerodynamic theory is employed.

The resulting equations of motion are

$$[B]\{\dot{X}(\Psi)\} = [A(\Psi)]\{X(\Psi)\} + \{f(\Psi)\} + \{F(\Psi, \{X\}, \{\dot{X}\})\} \tag{A3}$$

where $\{X(\Psi)\} \equiv (\dot{\beta}, \beta, \dot{\zeta}, \zeta, \dot{\theta}_e, \theta_e)^T$

The matrices $[A]$ and $[B]$ and vectors $\{f\}$ and $\{F\}$ are given in Ref. 24.

The blade perturbation equations of motions about the trim state are obtained by substituting $\Delta\{X(\Psi)\} = \{X(\Psi)\} - \{X_0(\Psi)\}$ into Eq. (A3) where $\{X_0(\Psi)\} = (\dot\beta_0, \beta_0, \dot\zeta_0, \zeta_0, \dot\theta_{e_0}, \theta_{e_0})^T$. The resulting perturbation equation is given by

$$\Delta\{\dot X\} = [D(\Psi)]\Delta\{X\} \tag{A4}$$

where

$$[D(\Psi)] = \left([B] + \left[B(\{X_0\})\right]^*\right)^{-1}\left([A(\Psi)] + \left[A(\Psi, \{X_0\})\right]^*\right)$$

Again, the matrices $[A]^*$ and $[B]^*$ are given in Ref. 24.

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